MATH 315 TEST 2 (SAMPLE EXAM QUESTIONS)

NOTE: THERE WILL BE FEWER QUESTIONS ON THE EXAM.
Calculators are NOT allowed for this examination. Show all work and justify your answers.
Credit will be based primarily on your work and explanations, not just the final answer.

1. Find the general solution for the differential equation \( y'' + 3y' + 2y = \cos(t) \).

2. Find the general solution for the differential equation \( y'' - 2y' + y = e^t \).

3. Use the method of undetermined coefficients to find a particular solution for the following differential equation:
   \[ y'' - y = 3t^2. \]

4. For the following differential equation
   \[ y'' - 4y' + 4y = e^{3t} \]
the homogeneous solution is given by
   \[ y_h(t) = C_1 e^{2t} + C_2 t e^{2t} \]

   (a) Use the method of variation of parameters to find a particular solution. \textit{Note: No points if the method of variation of parameters is not used.}

   (b) Find the general solution for the differential equation.

5. Find the Laplace transform of the following functions:
   (a) \( 1 + e^{-t} \)
   (b) \( 3 + t + e^{-t} \sin 2t \)
   (c) \( e^t H(t - 2) \)
   (d) \( e^{-t+1} H(t - 1) \)

6. Find the inverse Laplace transform of the following functions:
   (a) \( \frac{5}{s^2 + 3} \)
   (b) \( \frac{1}{(s - 2)^4} \)
   (c) \( \frac{1}{s^2 + 3s} \)
   (d) \( \frac{s + 1}{s^2 + s - 2} \)
   (e) \( \frac{e^{-4s}}{s + 4} \)
7. Find the Laplace transform \( Y(s) \) for the solution \( y(t) \) to the equation

\[
y' - 2y = \begin{cases} 
t, & 0 \leq t < 1 \\
0, & t \geq 1
\end{cases}
\]

with initial value \( y(0) = -1 \). (note: you do NOT need to find \( y(t) \)).

8. Use Laplace transforms to solve the equation

\[
y'' + 4y = e^{-t}
\]

with initial values \( y(0) = 0, \ y'(0) = 0 \).

9. Use the Laplace transform to solve the following initial value problem:

\[
y'' + 4y = 3\delta(t - \pi), \quad y(0) = 0, \ y'(0) = 0.
\]

10. Consider the equation \( y'' + xy' + 2y = 0 \).

(a) Explain why \( x_0 = 0 \) is an ordinary point for the equation.

(b) Find the recursion formula for the coefficients \( a_n \) in the power series representation of the solution \( y(x) = \sum_{n=0}^{\infty} a_n x^n \).

(c) Determine the first three nonzero terms of two linearly independent solutions.

11. Consider the equation

\[
y'' + 4x^2y = 0, \quad y(0) = 1, \ y'(0) = 1.
\]

Assume a series solution of the form

\[
y = \sum_{n=0}^{\infty} a_n x^n
\]

(a) Explain why \( x = 0 \) is an ordinary point.

(b) Find the first five nonzero terms.
Solutions

1. \( y(t) = C_1e^{-2t} + C_2e^{-t} + \frac{3}{10}\sin(t) + \frac{1}{10}\cos(t) \)

2. \( C_1e^t + C_2te^t + \frac{1}{2}t^2e^t \)

3. \( y_p = -3t^2 - 6 \)

4a. \( y_p(t) = [-te^t + e^t]e^{2t} + [e^t]te^{2t} = e^{3t} \). This took some work!

4b. \( y(t) = C_1e^{2t} + C_2e^{-t} + e^{3t} \)

5a. \( \frac{1}{s} + \frac{1}{s+1} \)

5b. \( \frac{3}{s} + \frac{1}{s^2} + \frac{2}{(s+1)^2 + 4} \)

5c. \( \frac{e^{2-2s}}{s-1} \)

5d. \( \frac{e^{-s}}{s+1} \)

6a. \( \frac{5\sqrt{3}}{3} \sin(\sqrt{3}t) \)

6b. \( \frac{1}{6}t^3e^{2t} \)

6c. \( \frac{1}{3}[1 - e^{-3t}] \)

6d. \( \frac{1}{3}e^{-2t} + \frac{2}{3}e^t \)

6d. \( e^{-4(t-4)}H(t-4) \)

6f. \( H(t-1)[1 - e^{1-t}] \)

7. \( Y(s) = \frac{1}{s^2(s-2)} - \frac{e^{-s}}{s^2(s-2)} - \frac{e^{-s}}{s(s-2)} - \frac{1}{s-2} \)

8. \( y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}\cos(2t) + \frac{1}{10}\sin(2t) \)

9. \( y(t) = \frac{3}{2}H(t-\pi)\sin(2t-2\pi) \)

10a. Since the functions \( x \) and 2 are polynomials, every (finite) point is an analytic point with an infinite radius of convergence. In particular \( x_0 = 0 \) is an analytic point of both \( x \) and 2 and hence it is an ordinary point of the equation.

10b. \( a_0 \) and \( a_1 \) are arbitrary, and \( a_{n+2} = -\frac{a_n}{(n+1)} \)

10c. For \( y_1(x) \) take \( y_1(0) = 1 \) and \( y_1'(0) = 0 \) so that \( a_0 = 1 \) and \( a_1 = 0 \). For \( y_2(x) \) take \( y_2(0) = 0 \) and \( y_2'(0) = 1 \) so that \( a_0 = 0 \) and \( a_1 = 1 \). Then

\[ y_1(t) = 1 - x^2 + \frac{1}{3}x^4 - ... \]

\[ y_2(t) = x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - ... \]

Checking the Wronskian evaluated at \( x = 0 \) shows that these two functions are linearly independent.

11a. Since the functions 0 and \( 4x^2 \) are polynomials, every (finite) point is an analytic point with an infinite radius of convergence. In particular \( x_0 = 0 \) is an analytic point of both 0 and \( 4x^2 \) and hence it is an ordinary point of the equation.

11b. \( a_0 = 1 \) and \( a_1 = 1 \) from the given initial conditions, and so we have: \( a_0 = a_1 = 1, a_4 = -\frac{1}{3}, a_5 = -\frac{1}{5}, a_8 = \frac{1}{42} \).