9.1 Parametric Curves

So far we have discussed equations in the form \( y = f(x) \). Sometimes \( x \) and \( y \) are given as functions of a parameter.

**Example.** Projectile Motion

\( \langle \text{Sketch } x \text{ and } y \text{ axes, cannon at origin, trajectory} \rangle \)

Mechanics gives \( x(t) \) and \( y(t) \). Time \( t \) is a parameter. ■

Given parameter \( t \). Then

\[
\begin{align*}
x &= f(t), \quad y = g(t)
\end{align*}
\]

are parametric equations for a curve in the \( xy \)-plane.

**Example.** \( x = 2 - t, \quad y = 3 - 2t \)

Draw the curve in the \( xy \)-plane.

\[
\begin{array}{ccc}
t & x & y \\
0 & 2 & 3 \\
1 & 1 & 1 \\
2 & 0 & -1
\end{array}
\]

\( \langle \text{Sketch axes and line in } xy \text{-plane} \rangle \)

Eliminate \( t \)

\[
\begin{align*}
t &= 2 - x \\
y &= 3 - 2(2 - x) \\
y &= 2x - 1
\end{align*}
\]

**Example** \( x = r \cos(\theta), \quad y = r \sin(\theta) \) where \( 0 \leq \theta \leq \pi/2 \)

\[
\begin{array}{ccc}
\theta & x & y \\
0 & r & 0 \\
\pi/6 & \frac{\sqrt{3}}{2} r & \frac{1}{2} r \\
\pi/4 & \frac{\sqrt{2}}{2} r & \frac{\sqrt{2}}{2} r
\end{array}
\]
\[
\frac{\pi}{3} \quad \frac{1}{2}r \quad \frac{\sqrt{3}}{2}r \\
\frac{\pi}{2} \quad 0 \quad \frac{r}{r}
\]

\[\langle 0 \ldots r \ldots x-, 0 \ldots r \ldots y-, \text{plot points, draw portion of circle with arrow, show } \theta \rangle\]

Parameter is the angle \( \theta \).

Eliminate \( \theta \)

\[\cos(\theta) = \frac{x}{r}, \quad \sin(\theta) = \frac{y}{r}\]

\[
\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1
\]

Gives the first quadrant portion of a circle of radius \( r \). ■

**Example.** \( x = \tan(\theta), \quad y = \sec(\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \)

\[
\tan^2(\theta) + 1 = \sec^2(\theta)
\]

\[x^2 + 1 = y^2\]

\[y^2 - x^2 = 1\]

hyperbola with asymptotes \( y = \pm x \)

\[\langle \text{sketch } xy\text{-axes, asymptotes, hyperbola} \rangle\]

\[-\frac{\pi}{2} < \theta < \frac{\pi}{2}\]

\[-\infty < x = \tan(\theta) < \infty\]

\[y = \sec(\theta) > 0\]

parametric equations describe the top branch of the hyperbola ■

A **cycloid** is a curve traced by a point on the rim of a rolling wheel.

\[\langle \text{sketch wheel, wheel rolled about a quarter turn ahead, portion of cycloid} \rangle\]

Find parametric equations
circle has radius $r$

$P(x, y)$ a point on the cycloid

$|OT| = r\theta$ length of arc $PT$

$|PQ| = r \sin (\theta)$

$|QC| = r \cos(\theta)$

$x = |OT| - |PQ| = r(\theta - \sin(\theta))$

$y = |TC| - |QC| = r(1 - \cos(\theta))$

Parametric equations for cycloid

$x = r(\theta - \sin(\theta))$

$y = r(1 - \cos(\theta))$

Table

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\pi r$</td>
<td>$2r$</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>$2\pi r$</td>
<td>0</td>
</tr>
</tbody>
</table>

(sketch xy-axes, plot points, draw curve through them)

one arch of the cycloid

**Drawing Graphs of Parametric Equations using Maple**

Command form: plot([x-expression, y-expression, parameter range],scaling=constrained);

(show graph of parametric equations)
9.2 Calculus with parametric curves

Tangents

Curve \( C \) in \( xy \)-plane described by parametric equations

\[
x = f(t)
\]
\[
y = g(t)
\]

Chain rule

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}
\]

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}
\]

This gives the slope of curve \( C \)

Let \( y' = dy/dx \)

\[
\frac{dy'}{dx} = \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}
\]

This gives the concavity of the curve \( C \)

Example  \( x = \tan(\theta), \ y = \sec(\theta), \ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \)

(a) Find the equation of the line tangent to the curve at \( \theta = \pi/4 \).

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sec(\theta)\tan(\theta)}{\sec^2(\theta)} = \frac{\tan(\theta)}{\sec(\theta)} = \sin(\theta)
\]

At \( \theta = \frac{\pi}{4} \)

\[
x_0 = \tan\left(\frac{\pi}{4}\right) = 1
\]

\[
y_0 = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}
\]

\[
m = \frac{dy}{dx} = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}
\]

equation of tangent line
(y - y₀) = m(x - x₀)

y - \sqrt{2} = \frac{1}{\sqrt{2}} (x - 1)

or

y = \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}}

(b) Sketch the curve and the tangent line

\tan^2(\theta) + 1 = \sec^2(\theta)

x^2 + 1 = y^2

y^2 - x^2 = 1

<table>
<thead>
<tr>
<th>\theta</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>\frac{\pi}{4}</td>
<td>1</td>
<td>\sqrt{2}</td>
</tr>
<tr>
<td>\frac{\pi}{4}</td>
<td>-1</td>
<td>\sqrt{2}</td>
</tr>
</tbody>
</table>

(sketch xy-axes, asymptotes, plot points, draw upper branch of hyperbola, sketch tangent line)

\frac{dy}{dx} = \sin(\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}

(c) Find \frac{d^2y}{dx^2} and discuss the concavity of the curve.

Since y' = \sin(\theta)

\frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta} = \frac{\cos(\theta)}{\sec^2(\theta)} = \cos^3(\theta) > 0, \quad \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}

The top branch of the hyperbola is concave up.

Example. Cycloid

x = r(\theta - \sin(\theta)),

y = r(1 - \cos(\theta))

Table with additional row

<table>
<thead>
<tr>
<th>\theta</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
</table>
Math 172 Chapter 9A notes

(a) Discuss the slope at $\theta = \pi, \pi/2$ and in the limit as $\theta \to 0^+$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \sin(\theta)}{r(1-\cos(\theta))} = \frac{\sin(\theta)}{1-\cos(\theta)}$$

At $\theta = \pi$

$$\frac{dy}{dx} = \frac{\sin(\pi)}{1-\cos(\pi)} = 0 \quad \langle \text{add segment with slope 0} \rangle$$

At $\theta = \pi/2$

$$\frac{dy}{dx} = \frac{\sin(\pi/2)}{1-\cos(\pi/2)} = 1 \quad \langle \text{add segment with slope 1} \rangle$$

As $\theta \to 0^+$

$$\lim_{\theta \to 0^+} \frac{dy}{dx} = \lim_{\theta \to 0^+} \frac{\sin(\theta)}{1-\cos(\theta)} = \lim_{\theta \to 0^+} \frac{\cos(\theta) - 1}{\sin(\theta)}$$

$$= \frac{\cos(\theta) - 1}{1-\cos(\theta)^2}$$

where L'Hospital's rule has been used.

The cycloid has a vertical tangent in the limit $\theta \to 0^+$. $\langle \text{add} \rangle$

(b) Discuss concavity of the cycloid

$$\frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta} \frac{\sin(\theta)}{1-\cos(\theta)} = \frac{\cos(\theta)(1-\cos(\theta)) - \sin(\theta) \sin(\theta)}{(1-\cos(\theta))^2}$$

$$= \frac{\cos(\theta) - 1}{(1-\cos(\theta))^2}$$

$$= \frac{-1}{1-\cos(\theta)}$$

$$\frac{dx}{d\theta} = r(1 - \cos(\theta))$$

$$\frac{d^2y}{dx^2} = \frac{-1}{r(1-\cos(\theta))^2}$$
\[ \frac{d^2 y}{dx^2} < 0 \text{ for } 0 < \theta < 2\pi \]

The cycloid is concave down over the entire arch, except for the cusp points \( \theta = 0, \pi, \ldots \) where it is not defined.

**Areas**

\( \langle \text{sketch } 0 \ldots a \ldots b \ldots x-, 0 \ldots, \text{ curve } y \text{ above } [a, b] \rangle \)

Area under the curve

\[ A = \int_a^b y \, dx \]

Suppose

\[ x = f(t), \ y = g(t), \ \alpha \leq t \leq \beta, \text{ where } a = f(\alpha) \text{ and } b = f(\beta) \]

\[ A = \int_\alpha^\beta g(t)f'(t) \, dt \]

**Example.** Find the area of the circle

\[ x = \cos(t), \ y = \sin(t), \ 0 \leq t \leq 2\pi \]

\( \langle \text{sketch } xy\text{-axes, unit circle, angle } t \text{ CCW from positive } x\text{-axis} \rangle \)

\[ A = 2 \int_{-1}^{1} y \, dx \]

\[ = 2 \int_0^\pi \sin(t) \ (-) \sin(t) \, dt \]

\[ = 2 \int_0^\pi \sin^2(t) \, dt \]

\[ = 2 \int_0^{\pi/2} (1 - \cos(2t)) \, dt \]

\[ = \pi - \int_0^\pi \cos(2t) \, dt \]

\[ = \pi \]

Notice the last integral integrates over a full period of cosine.

**Example.** Find the area of the asteroid
\( x = \cos^3(t) \), \( y = \sin^3(t) \) \( 0 \leq t \leq 2\pi \)

(sketch \(-1 \ldots 0 \ldots 1 \ldots x, \ldots -1 \ldots 0 \ldots 1 \ldots y, \) astroid)

\[
A = 2 \int_0^\pi \sin^3(t)3 \cos^2(t)(-\sin(t)) \, dt
\]

\[
= 6 \int_0^\pi \sin^4(t) \cos^2(t) \, dt
\]

\[
\sin^2(t) = \frac{1}{2}(1 - \cos(2t))
\]

\[
\sin^4(t) = \frac{1}{8}(1 - \cos(2t))^2
\]

\[
\cos^2(t) = \frac{1}{2}(1 + \cos(2t))
\]

therefore

\[
\sin^4(t) \cos^2(t) = \frac{1}{8}(1 - \cos(2t))(1 + \cos(2t))
\]

\[
= \frac{1}{8}(1 - \cos(2t)(1 - \cos^2(2t))
\]

\[
= \frac{1}{8}(1 - \cos(2t) - \cos^2(2t) + \cos^3(2t))
\]

so

\[
A = \frac{3}{4} \int_0^\pi (1 - \cos(2t) - \cos^2(2t) + \cos^3(2t)) \, dt
\]

Notice

\[
\int_0^\pi \cos(2t) \, dt = 0
\]

\[
\int_0^\pi \cos^3(2t) \, dt = 0
\]

\[
\int_0^\pi \cos^2(2t) \, dt = \pi/2
\]

The first two integrals are seen to be zero by symmetry because the integrands are odd powers of cosine and the argument \(2t\) varies over a full period.

The value of the last integral can be seen from the fact that the average value of \(\sin^2\) or \(\cos^2\) over their period (\(\pi\)) is \(\frac{1}{2}\). It is also an immediate consequence of the half angle identities.

\[
A = \frac{3}{4} \left( \pi - 0 - \frac{\pi}{2} + 0 \right) = \frac{3}{8} \pi
\]
**Arc Length**

Symbolically \( L = \int_C ds \)

\( \langle 0 \ldots a \ldots b \ldots x-, 0 \ldots y-, \text{ curve } C \text{ over } [a, b], \text{ triangle } dx, dy, ds \rangle \)

\[ L = \int_c \sqrt{dx^2 + dy^2} \]

Suppose \( C \) is described by parametric equations

\[ x = f(t), \quad y = g(t) \]

\[ dx = \frac{dx}{dt} \, dt, \quad dy = \frac{dy}{dt} \, dt \]

then

\[ L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

where \( a = f(\alpha) \) and \( b = f(\beta) \).

**Example** Find the length of the curve

\[ x = e^t - t, \quad y = 4e^{\frac{t}{2}}, \quad 0 \leq t \leq 1 \]

\[ \frac{dx}{dt} = e^t - 1, \quad \frac{dy}{dt} = 2e^{\frac{t}{2}} \]

\[ \left(\frac{dx}{dt}\right)^2 = e^{2t} - 2e^t + 1, \quad \left(\frac{dy}{dt}\right)^2 = 4e^t \]

\[ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2 \]

\[ L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

\[ = \int_0^1 (e^t + 1) \, dt \]

\[ = e^t + t \bigg|_0^1 \]

\[ = (e + 1) - (1 + 0) \]

\[ = e \quad \blacksquare \]

**Example** Find the total length of the astroid
\[ x = a \cos^3(\theta), \quad y = a \sin^3(\theta) \]

\[ \langle \ldots a\ldots a\ldots x\ldots \ldots a\ldots a\ldots y\ldots , \text{astroid} \rangle \]

\[ \frac{dx}{d\theta} = a3 \cos^2(\theta)(-\sin(\theta)) = -3 \cos^2(\theta)\sin(\theta) \]

\[ \left(\frac{dx}{d\theta}\right)^2 = 9a^2 \cos^4(\theta) \sin^2(\theta) \]

\[ \frac{dy}{d\theta} = a3 \sin^2(\theta)\cos(\theta) \]

\[ \left(\frac{dy}{d\theta}\right)^2 = 9a^2 \sin^4(\theta) \cos^2(\theta) \]

\[ L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta \]

\[ \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2 \cos^2(\theta) \sin^2(\theta)(\cos^2(\theta) + \sin^2(\theta)) \]

\[ = 9a^2 \cos^2(\theta) \sin^2(\theta) \]

For \( 0 \leq \theta \leq \pi/2 \)

\[ \cos(\theta) \geq 0 \text{ and } \sin(\theta) \geq 0 \]

Therefore

\[ L = 4 \int_0^{\pi/2} 3a \cos(\theta) \sin(\theta) \, d\theta \]

\[ = 12a \int_0^{\pi/2} \frac{1}{2} \sin(2\theta) \, d\theta \]

where we used \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \)

\[ L = 6a \left( -\frac{1}{2} \cos(2\theta) \right) \bigg|_0^{\pi} = 6a \left( -\frac{1}{2} \right) (-1 - 1) = 6a \]

\[ \boxed{9.3 \text{ Polar Coordinates}} \]

Cartesian or rectangular coordinates

\[ \langle \ldots \ldots x \ldots \ldots \ldots y\ldots , \ P(x,y) \rangle \]

\( P \) is associated with a unique ordered pair \((x,y)\)
Polar Coordinates

\( \langle \ldots 0 \ldots , \ldots 0 \ldots , \text{indicate the polar axis, ray, } r, \theta, P \rangle \)

- \( r \) distance from origin
- \( \theta \) angle measured CCW from the polar axis

**Example.** Plot \( Q: (r, \theta) = (2, \frac{5\pi}{6}) \)

\( \langle \ldots 0 \ldots , \ldots 0 \ldots , \text{ray, } \theta = \frac{5\pi}{6}, \ r = 2 \rangle \)

Representation in polar coordinates is **not unique**.

\( \langle \text{axes, } P, r, \theta, \theta + \pi, \text{indicate backward extension of ray through origin} \rangle \)

\( P \) may be represented by

\( (r, \theta) \)

\( (r, \theta + 2\pi), \ (r, \theta + 4\pi), \ldots \)

\( (r, \theta - 2\pi), \ (r, \theta - 4\pi), \ldots \)

\( (-r, \theta + \pi) \)

**Example.** May represent \( Q: \left( 2, \frac{5\pi}{6} \right) \) by \( \left( 2, -\frac{7\pi}{6} \right) \)

Sometimes restrict

\( r \geq 0, \ 0 \leq \theta < 2\pi \)

every point except \( r = 0 \) has a unique representation.

\( r = 0, \ \theta = \text{anything} \)

always represents the origin

---

**Conversion from polar to Cartesian coordinates**

\( \langle \ldots 0 \ldots , x \ldots , \ldots 0 \ldots , y \ldots , \text{ray, } r, \theta, \text{projection from tip to } x\text{-axis} \rangle \)
Example. Convert $P: (r, \theta) = (2, \frac{5\pi}{6})$ to Cartesian coordinates

\[
x = r \cos(\theta) = 2 \cos \left(\frac{5\pi}{6}\right) = 2 \left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}
\]
\[
y = \sin(\theta) = 2 \sin \left(\frac{5\pi}{6}\right) = 2 \left(\frac{1}{2}\right) = 1 \quad \blacksquare
\]

Conversion from Cartesian to polar coordinates

\[
x = r \cos(\theta)
\]
\[
y = r \sin(\theta)
\]
\[
x^2 + y^2 = r^2 (\cos^2(\theta) + \sin^2(\theta)) = r^2
\]
\[
y = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)
\]
\[
r^2 = x^2 + y^2
\]
\[
\tan(\theta) = \frac{y}{x}
\]

Example. Convert $Q: (x, y) = (1,1)$ to polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$.

\[
\langle \ldots 0 \ldots 1 \ldots, \ldots 0 \ldots 1 \ldots, Q \rangle
\]
\[
r^2 = x^2 + y^2 = 2 \quad \quad \quad \quad \quad \quad [1a]
\]
\[
\tan(\theta) = 1 \quad \quad \quad \quad \quad \quad [1b]
\]

Solutions of [1] with $r \geq 0$ and $0 \leq \theta < 2\pi$:

\[
r = \sqrt{2}, \quad \quad \theta = \frac{\pi}{4}, \quad \frac{5\pi}{4}
\]

Notice that $\tan(\theta + \pi) = \frac{\sin(\theta + \pi)}{\cos(\theta + \pi)} = \frac{-\sin(\theta)}{-\cos(\theta)} = \tan(\theta)$

But $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$ is not the same point as $(\sqrt{2}, \frac{5\pi}{4})$

Equations [1] are not sufficient, we must also choose $\theta$ to be in the correct quadrant. $\blacksquare$
Polar Equations

General form

\[ F(r, \theta) = 0 \]

Common form

\[ r = f(\theta) \]

**Example.** \( r = a \)

\langle \text{axes, circle of radius } a \rangle

circle, center at origin, with radius \( a \)

To find equation in Cartesian coordinates, square both sides: \( r^2 = a^2 \)

giving \( x^2 + y^2 = a^2 \)  ■

**Example.** Find the polar equation for the curve represented by

\[ x^2 + y^2 = ay \]  \[ \text{[2]} \]

Let \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \), then \( x^2 + y^2 = r^2 \)

Eq. [2] becomes

\[ r^2 = ar \sin(\theta) \]

Solutions are \( r = 0 \) or

\[ r = a \sin(\theta) \]

[2] is an equation for a circle. To see, complete squares

\[ x^2 + y^2 - ay = 0 \]

\[ x^2 + y^2 - ay + \left( \frac{a^2}{4} - \frac{a^2}{4} \right) = 0 \]

\[ x^2 + \left( y - \frac{a}{2} \right)^2 = \left( \frac{a}{2} \right)^2 \]

\langle \text{sketch axes, circle centered at } \left(0, \frac{a}{2} \right) \text{ with radius } \frac{a}{2} \rangle

circle with radius \( a/2 \) and center \( \left(0, \frac{a}{2} \right) \). ■
Symmetry of solutions of \( F(r, \theta) = 0 \)

(1) If \( F(r, \pi - \theta) = 0 \) whenever \( F(r, \theta) = 0 \), the solution is symmetric about the \( y \)-axis.

(2) If \( F(r, -\theta) = 0 \) whenever \( F(r, \theta) = 0 \), the solution is symmetric about the \( x \)-axis.

(3) If \( F(-r, \theta) = 0 \) whenever \( F(r, \theta) = 0 \), the solution is symmetric about the origin.

Example. \( r = a \sin (\theta) \).

\[
F(r, \theta) = r - a \sin (\theta) = 0.
\]

\[
\sin(\pi - \theta) = \sin (\theta)
\]

\[
F(r, \pi - \theta) = F(r, \theta)
\]

Solution is symmetric about the \( y \)-axis.

Example. \( r = 1 - \cos(\theta) \).

\[
\cos(-\theta) = \cos(\theta)
\]

Solution is symmetric about the \( x \)-axis.

Sketch \( r = 1 - \cos (\theta) \)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>( 1 + \sqrt{2}/2 \approx 1.7 )</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>1</td>
</tr>
<tr>
<td>( 3\pi/4 )</td>
<td>( 1 - \sqrt{2}/2 \approx 0.3 )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( 1 - 1 = 0 )</td>
</tr>
</tbody>
</table>

Sketch points in first quadrant, draw smooth curve through them, complete in fourth quadrant.

This is a cardioid.

Tangents to Polar Curves

Common form of a polar equation
\( r = f(\theta) \)

where \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \).

Consider \( \theta \) as a parameter, then from the results of section 9.2

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr}{d\theta} \frac{\sin(\theta) + r \cos(\theta)}{dr \cos(\theta) - r \sin(\theta)}
\]

Let \( r = 0 \)

\[
\frac{dy}{dx} = \tan(\theta)
\]

Suppose the graph of \( r = f(\theta) \) passes through the origin at an angle \( \theta \)

\( \langle \text{axes, ray } r = \theta, \text{ curve passing through origin tangent to ray} \rangle \)

slope = \( \tan(\theta) \) \hspace{1cm} \langle \text{add projection down from ray to } x\text{-axis to complete a right triangle} \rangle

**Example.** Find the slope of the line tangent to

\[ r = \sin(3\theta) \]

at \( \theta = \pi/6 \).

\[ y = r \sin(\theta) = \sin(3\theta) \sin(\theta) \]

\[
\frac{dy}{d\theta} = 3 \cos(3\theta) \sin(\theta) + \sin(3\theta) \cos(\theta)
\]

\[ x = r \cos(\theta) = \sin(3\theta) \cos(\theta) \]

\[
\frac{dx}{d\theta} = 3 \cos(3\theta) \cos(\theta) - \sin(3\theta) \sin(\theta)
\]

At \( \theta = \pi/6, \ 3\theta = \pi/2 \)

\[ \sin(3\theta) = 1, \ \cos(3\theta) = 0, \ \sin(\theta) = 1/2, \ \cos(\theta) = \sqrt{3}/2 \]

\[
\left. \frac{dy}{d\theta} \right|_{\pi/6} = 3 \cdot 0 \cdot \frac{1}{2} + 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}
\]

\[
\left. \frac{dx}{d\theta} \right|_{\pi/6} = 3 \cdot 0 \cdot \frac{\sqrt{3}}{2} - 1 \cdot \frac{1}{2} = -\frac{1}{2}
\]

Thus \( \left. \frac{dy}{dx} \right|_{\pi/6} = \frac{\left. \frac{dy}{d\theta} \right|_{\pi/6}}{\left. \frac{dx}{d\theta} \right|_{\pi/6}} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3} \)
Sketch \( r = \sin(3\theta) \)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( 3\theta )</th>
<th>( r = \sin(3\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \pi/12 )</td>
<td>( \pi/4 )</td>
<td>( \sqrt{2}/2 )</td>
</tr>
<tr>
<td>( \pi/6 )</td>
<td>( \pi/2 )</td>
<td>1</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>( 3\pi/4 )</td>
<td>( \sqrt{2}/2 )</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>( \pi )</td>
<td>0</td>
</tr>
</tbody>
</table>

(sketch axes, ray \( \theta = \pi/3 \), plot points, draw curve)

### 9.4 Areas and Lengths in Polar Coordinates

**Area of a sector of a circle**

\[
\frac{A}{\pi r^2} = \frac{\theta}{2\pi}
\]

\[A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta\]

**Area bounded by a polar curve \( r = f(\theta) \):**

\(0\), dashed polar axis, ray \( \theta = a\), ray \( \theta = b\), curve \( r = f(\theta) \) between these angles, small sector subtended by \( d\theta \), its angle \( \theta \)

Area of slice with angle \( d\theta \): \[
\frac{1}{2} r^2 d\theta = \frac{1}{2} f(\theta)^2 d\theta
\]

Total area bounded by \( f(\theta) \) and the rays \( \theta = a \) and \( \theta = b \)

\[A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta\]

**Example.** Find the area enclosed by the loop of \( r = \sin(3\theta) \) between \( \theta = 0 \) and \( \theta = \pi/3 \).

\[A = \frac{1}{2} \int_0^{\pi/3} \sin^2(3\theta) \ d\theta\]

Use the half-angle identity \( \sin^2(3\theta) = \frac{1}{2}(1 - \cos(6\theta)) \)

\[A = \frac{1}{4} \int_0^{\pi/3} (1 - \cos(6\theta)) \ d\theta = \frac{\pi}{12} - \frac{1}{4} \int_0^{\pi/3} \cos(6\theta) \ d\theta\]
\[= \frac{\pi}{12} \left( -\frac{1}{4} \int_0^{2\pi} \cos(u) \frac{1}{6} \, du \right)\]
\[= \frac{\pi}{12}\]

with the substitution \( u = 6\theta \).

**Example.** Find the area of the region that lies inside the graph of
\[r = 1 + \cos(\theta) = f(\theta)\]
but outside the graph of
\[r = 3 \cos(\theta) = g(\theta)\]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \cos(\theta) )</th>
<th>( 1 + \cos(\theta) )</th>
<th>( 3 \cos(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>0.7</td>
<td>1.7</td>
<td>2.1</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>0.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( 2\pi/3 )</td>
<td>-0.5</td>
<td>0.5</td>
<td>-1.5</td>
</tr>
<tr>
<td>( 3\pi/4 )</td>
<td>-0.7</td>
<td>0.3</td>
<td>-2.1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>-1</td>
<td>0</td>
<td>-3</td>
</tr>
</tbody>
</table>

\(<\ldots 0 \ldots 1 \ldots 2 \ldots 3 \ldots, \ldots -1 \ldots 0 \ldots 1 \ldots,\ldots,\)\) plot points for cardioids and draw curve (lower half plane by symmetry), plot points for circle and draw curve. Area \( A \) has components in the first and second quadrants.

Area between curves is \( 2A \).

Find intersections between curves
\[1 + \cos(\theta) = 3 \cos(\theta)\]
\[1 = 2 \cos(\theta)\]
\[\frac{1}{2} = \cos(\theta)\]
\[\theta = \pm \frac{\pi}{3}\]

This misses the intersection at the origin! Why? Graphs have different values of \( \theta \) at the origin.

\[A = \int_{\pi/3}^{\pi} f(\theta)^2 \, d\theta - \int_{\pi/3}^{\pi/2} g(\theta)^2 \, d\theta\]

\[f(\theta)^2 = (1 + \cos(\theta))^2 = 1 + 2 \cos(\theta) + \cos^2(\theta)\]

\[= 1 + 2 \cos(\theta) + \frac{1}{2}(1 + \cos(2\theta))\]
\[ f(\theta) = \frac{3}{2} + 2\cos(\theta) + \frac{1}{2}\cos(2\theta) \]

\[
\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} f(\theta)^2 d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left( \frac{3}{4} \pi \right) + \frac{1}{8}\sin(2\theta) \mid_{\frac{\pi}{3}}^{\frac{\pi}{2}}
\]

\[
= \frac{3}{4} \left( \pi - \frac{\pi}{3} \right) + \frac{1}{8} \left( \sin(2\theta) \right) \mid_{\frac{\pi}{3}}^{\frac{\pi}{2}}
\]

\[
= \frac{3}{4} \left( \frac{2\pi}{3} \right) + \frac{1}{8} \left( 0 - \frac{\sqrt{3}}{2} \right)
\]

\[
= \frac{\pi}{2} - \frac{\sqrt{3}}{2} \cdot \frac{9}{8}
\]

\[ g(\theta)^2 = 9\cos^2(\theta) = \frac{9}{2} (1 + \cos(2\theta)) \]

\[
\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} g(\theta)^2 d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{9}{4} (1 + \cos(2\theta)) d\theta
\]

\[
= \frac{9}{4} \left( \pi - \frac{\pi}{3} \right) + \frac{9}{8} \left( \frac{1}{2}\sin(2\theta) \right) \mid_{\frac{\pi}{3}}^{\frac{\pi}{2}}
\]

\[
= \frac{9}{4} \cdot \frac{1}{2}\pi + \frac{9}{8} \left( 0 - \frac{\sqrt{3}}{2} \right)
\]

\[
= \frac{3}{8} \pi - \frac{\sqrt{3}}{2} \cdot \frac{9}{8}
\]

Then \[ A = \frac{\pi}{8} \] Area between the curves \[ 2A = \frac{\pi}{4} \]

**Arc Lengths in Polar Coordinates**

\( \langle \text{sketch 0}, 0, \ldots, \text{ray } \theta = a, \text{ray } \theta = b, \text{curve } C \rangle \)

Symbolically

\[ L = \int_L ds = \int_C \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} \ d\theta \]

where

\[ x = r \cos(\theta), \quad y = \sin(\theta) \]

Keeping in mind that \( r \) depends on \( \theta \):

\[ \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos(\theta) - r \sin(\theta) \]

\[ \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin(\theta) + r \cos(\theta) \]
Thus

\[ L = \int_a^b \sqrt{\left( \frac{dr}{d\theta} \right)^2 + r^2} \ d\theta \]

**Example.** Find the length of the cardioid \( r = 1 + \cos(\theta) \).

\[
L = 2 \int_0^\pi \sqrt{\left( \frac{dr}{d\theta} \right)^2 + r^2} \ d\theta
\]

where

\[
\frac{dr}{d\theta} = -\sin(\theta)
\]

\[
r^2 + \left( \frac{dr}{d\theta} \right)^2 = (1 + \cos(\theta))^2 + \sin^2(\theta)
\]

\[
= 1 + 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)
\]

\[
= 2 + 2\cos(\theta)
\]

\[
L = 2 \int_0^\pi \sqrt{2 + 2\cos(\theta)} \ d\theta
\]

consider

\[
\cos^2 \left( \frac{\theta}{2} \right) = \frac{1}{2} (1 + \cos(\theta))
\]

\[
4 \cos^2 \left( \frac{\theta}{2} \right) = 2(1 + \cos(\theta))
\]

\[
\cos \left( \frac{\theta}{2} \right) \geq 0 \text{ for } 0 \leq \theta \leq \frac{\pi}{2}
\]

Then

\[
L = 2 \int_0^\pi 2 \cos \left( \frac{\theta}{2} \right) \ d\theta
\]
\[ = 4 \left( 2 \sin \left( \frac{\theta}{2} \right) \right) \bigg|_0^\pi \]

\[ = 8 \left( \sin \left( \frac{\pi}{2} \right) - \sin(0) \right) \]

\[ = 8 \quad \blacksquare \]