No books, notes or calculators. Please show your work and circle your answers.

1. The growth of cancerous tumors has been modeled by the Gompertz law

\[ \dot{N}(t) = -aN \ln(bN) \quad (1) \]

where \( N(t) \) is the number of cells in the tumor and \( a, b > 0 \) are parameters.

(a) (10 points) Remove the parameter \( a \) by introducing a dimensionless time \( \tau \). Use the chain rule to convert \( \dot{N} \) into \( N' \). Give the definition of \( \tau \) and the new differential equation obtained from the Gompertz equation.

\[
\tau = \alpha t \\
N' = \frac{dN}{d\tau} = \frac{dN}{dt} \frac{dt}{d\tau} = \alpha \frac{dN}{d\tau} = aN'
\]

From (1)

\[ aN' = -aN \ln(bN) \]

\[ N' = -N \ln(bN) \]

(b) (10 points) Remove the parameter \( b \) by introducing a new population variable \( x \) to obtain the differential equation

\[ x' = -x \ln(x) \]

Give the definition of \( x \).

\[ x = bN \quad \text{or} \quad N = \frac{x}{b} \]

\[ N' = \frac{1}{b} x' = -\frac{x}{b} \ln(x) \]

giving the desired form
2. Consider the 1D system: \( \frac{dx}{dt} = r - x - \frac{1}{x} = y_1 - y_2 \)

(a) (15 pts) Show that a bifurcation exists at a certain value \( r_c > 0 \) by plotting the phase portraits for \( r < r_c, r = r_c \) and \( r > r_c \) (don't worry about \( r < 0 \)). To plot the phase portraits, sketch \( y_1 = r - x \) versus \( y_2 = \frac{1}{x} \). Be sure to include arrows and fixed points on the horizontal axis, and indicate the stability of the fixed points.
2. The system \( \frac{dx}{dt} = r - x - \frac{1}{x} \) continued!

(b) (10 points) Find the values of \( r_c \) and \( x_c > 0 \) where the bifurcation occurs.

Tangency condition

1) \( y_1 = y_2 \) \( \Rightarrow r - x = \frac{1}{x} \)

2) \( y'_1 = y'_2 \) \( \Rightarrow -1 = -\frac{1}{x^2} \)

\( x_c = 1 \) From (1) \( r_c - 1 = 1 \) so \( r_c = 2 \)

(c) (10 points) Perform a linearized stability analysis of the system about an arbitrary fixed point \( x^* \). What are the conditions for \( x^* \) to be stable or unstable?

\[ f(x) = r - x - \frac{1}{x} \]
\[ f'(x) = -1 + x^{-2} \]

Stability for \( -1 + x^{-2} < 0 \)

\( x^{-2} < 1 \) \( \Rightarrow x^2 > 1 \)

Instability for \( x^2 < 1 \)
2. The system \( \frac{dx}{dt} = r - x - \frac{1}{x} \) continued!

(d) (15 pts) Using information from parts (a), (b) and (c), draw a bifurcation diagram for this system for parameter values \( r > 0 \). Indicate the stable and unstable branches. What is the name for this bifurcation?
3. Consider the system: $\frac{dx}{dt} = 4xy - 1$ and $\frac{dy}{dt} = 4x - y^2$.

(a) (10 points) Find the fixed point $(x^*, y^*)$.

(1) $4x^* y^* = 1$

(2) $y^2 = 4x^*$

Insert (2) into (1)

$y^3 = 1$

$x^* = 4^*, y^* = 1$

(b) (10 points) Calculate the Jacobian matrix $J$ for the system, evaluated at $(x^*, y^*)$.

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

$$J|_{(x^*, y^*)} = \begin{pmatrix} 4 & 1 \\ 4 & -2 \end{pmatrix}$$

$c) (10 points)$ Using the Jacobian, classify the fixed point using the classification of linear systems presented in class (and in chapter 5 of our text).

$$I = trace\left( J|_{(x^*, y^*)} \right) = 2$$

$$\Delta = \det\left( J|_{(x^*, y^*)} \right) = -8 - 4 = -12$$

The fixed point is a saddle since $\Delta < 0$.