Math 546  Assignment 3  due Thursday, September 23

1. Basic properties of the Kronecker product

(a) Let $X$, $Y$, $W$, and $Z$ be matrices whose dimensions are such that
the products $XW$ and $YZ$ are defined. Show that the product $(X \otimes Y)(W \otimes Z)$ is defined, and $(X \otimes Y)(W \otimes Z) = (XW) \otimes (YZ)$.

(Notice the following important special case. If $u$ and $v$ are column vectors such that $Xu$ and $Yv$ are defined, then $u \otimes v$ is a (long) column vector such that $(X \otimes Y)(u \otimes v) = Xu \otimes Yv$.)

(b) Suppose $X$ is $m \times m$ and $Y$ is $n \times n$. Suppose $\lambda$ is an eigenvalue of $X$ with associated eigenvector $u$, and $\mu$ is an eigenvalue of $Y$ with associated eigenvector $v$.

i. Show that $\lambda \mu$ is an eigenvalue of $X \otimes Y$ with associated eigenvector $u \otimes v$.

ii. Let $I_k$ denote the $k \times k$ identity matrix. Show that $\lambda + \mu$ is an eigenvalue of $X \otimes I_n + I_m \otimes Y$ with associated eigenvector $u \otimes v$.

(c) Show that if $u_1, \ldots, u_n$ is a set of $n$ orthonormal vectors, and $v_1, \ldots, v_m$ is a set of $m$ orthonormal vectors, then $u_i \otimes v_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, is a set of $nm$ orthonormal vectors.

2. Let $A$ denote the matrix of the linear system obtained by discretizing the two-dimensional Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega = (0, 1)^2, \quad u = v \text{ on } \partial \Omega$$

with mesh size $h = 1/n$ using the standard five-point stencil.

(a) Find a complete set of eigenvalues and eigenvectors of $A$.

(b) Prove that the finite difference scheme is stable, hence convergent of order 2.

3. Let $A$ denote the matrix of the linear system obtained by discretizing the three-dimensional Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega = (0, 1)^3, \quad u = v \text{ on } \partial \Omega$$

with mesh size $h = 1/n$ using the standard second-order finite-difference approximations to the second derivatives.
(a) Show that $A$ is block tridiagonal with blocks of dimension $(n - 1)^2 \times (n - 1)^2$ and that each of these blocks is itself block tridiagonal. Approximately what is the band width of $A$?

(b) Use tensor products to obtain a concise expression for $A$.
(Note: You can easily convince yourself that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, so the expression $X \otimes Y \otimes Z$ is unambiguous.)