Math 172  Exam #1  September 29, 2009

Name: Solutions  Section#:  
Signature:  ID#  

Instructions: Do all the problems. Circle your answers. To get credit, you need to show your work. The answer alone will not be counted as a solution.

1. (10 pts each) Evaluate each of the following integrals:

(a) \[ \int \frac{x}{x^2 - x - 2} \, dx = I \]

partial fraction decomposition

\[ \frac{x}{x^2 - x - 2} = \frac{x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \]

\[ x = A(x-2) + B(x+1) \]

\[ x = 2: \quad 2 = 3B \quad B = \frac{2}{3} \]

\[ x = -1: \quad -1 = -3A \quad A = \frac{1}{3} \]

Integrate

\[ I = \int \left( \frac{\frac{1}{3}}{x+1} + \frac{\frac{2}{3}}{x-2} \right) \, dx \]

\[ I = \frac{1}{3} \ln |x+1| + \frac{2}{3} \ln |x-2| + C \]
(b) \( \int \sqrt{1-x^2} \, dx \). Hint: Consider a substitution: \( x = \sin \theta \).

\[
I = \int \sqrt{1-\sin^2(\theta)} \cos \theta \, d\theta \\
= \int \cos^2(\theta) \, d\theta \\
= \int \frac{1}{2} (1 + \cos(2\theta)) \, d\theta \\
= \frac{1}{2} \theta + \frac{1}{2} \frac{\sin(2\theta)}{2} + C \\
= \frac{1}{2} \theta + \frac{1}{2} \sin(\theta) \cos(\theta) + C
\]

Triangle for \( x = \sin(\theta) \)

\[
I = \frac{1}{2} \sin^{-1}(x) + \frac{1}{2} x \sqrt{1-x^2} + C
\]

(c) \( \int \sin^3 x \cos x \, dx = I \)

Let \( u = \sin(x) \)
\[du = \cos(x) \, dx\]

\[
I = \int u^3 \, du \\
= \frac{1}{4} u^4 + C \\
I = \frac{1}{4} \sin^4(x) + C
\]
2. (10 pts each) Evaluate each of the following improper integrals:
(Hint: A limit relationship?)

(a) \[ \int_{1}^{\infty} x^2 \ln x \, dx = \lim_{t \to \infty} \int_{1}^{t} x^2 \ln \left( \frac{x}{2} \right) \, dx \]

but \[ x^2 \ln \left( \frac{x}{2} \right) > 1 \quad \text{for} \quad x > 3 \]

thus

\[ \lim_{t \to \infty} \int_{1}^{t} x^2 \ln \left( \frac{x}{2} \right) \, dx \geq \lim_{t \to \infty} \int_{3}^{t} 1 \, dx \]

\[ = \lim_{t \to \infty} (t - 3) \]

\[ = \infty \]

The improper integral diverges.
(b) $\int_0^1 x^{1/2} \, dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} x^{-1/2} \, dx$

$= \lim_{\varepsilon \to 0^+} 2 x^{1/2} \bigg|_{\varepsilon}^{1} = \lim_{\varepsilon \to 0^+} 2 (1 - \varepsilon^{1/2}) = 2$

3. (10pts) A force of 10 N is required to maintain a spring stretched from its natural length of 1m to a length of 2m. How much work is done in stretching the spring from 2m to 3m? (Hint: Hooke’s law: $f(x) = kx$.)
4. (10 pts) Find the area bounded between the curves \( y = \sqrt{x} \) and \( y = x^2 \).

\[
A = \int_0^1 (\sqrt{x} - x^2) \, dx
\]

\[
= \left[ \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \right]_0^1
\]

\[
= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
\]

5. (10 pts) Find the volume of the solid generated by rotating the region bounded by the curves \( y = \sqrt{x} \) and \( y = x^2 \) about the \( x \)-axis.

\[
V(x) = \pi \int_0^1 (\sqrt{x} - x^2) \, dx
\]

\[
= \pi \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1
\]

\[
= \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \pi \frac{5 - 2}{10}
\]

\[
= \pi \frac{2}{10}
\]
6. (10 pts) Use the method of cylindrical shells to find the volume of the solid generated by rotating the region bounded by the curves \( y = \sqrt{x} \) and \( y = x^2 \) about the \( y \)-axis.

\[
V = \int_0^1 2\pi y \left( y^{3/2} - y^2 \right) dy
= 2\pi \int_0^1 \left( y^{3/2} - y^3 \right) dy
= 2\pi \left[ \frac{2}{5} y^{5/2} - \frac{1}{4} y^4 \right]_0^1
= 2\pi \left( \frac{2}{5} - \frac{1}{4} \right)
= 2\pi \frac{8 - 5}{20} = \left[ \frac{3}{10} \right]
\]

7. (10 pts) Find the length of the curve: \( y = \frac{2}{3} (x - 2)^{3/2} \), \( 3 \leq x \leq 4 \).

\[ \textit{Not on Exam} \]