10.1 Three Dimensional Space

2D space

\[ \langle 0 \ldots x \ldots, 0 \ldots y \ldots, P(x, y) \rangle \] [Fig. 1]

Point \( P \) represented by \((x, y)\), an ordered pair of real nos.

Set of all ordered pairs

\[ \mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\} \]

Identify 2D space with \( \mathbb{R}^2 \)

3D space

\[ \langle 0 \ldots x \ldots, 0 \ldots y \ldots, 0 \ldots z \ldots, P(x, y, z) \rangle \] [Fig. 2]

Point \( P \) represented by \((x, y, z)\) an ordered triple of real nos.

Set of all ordered triples

\[ \mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\} \]

Identity 3D space with \( \mathbb{R}^3 \)

2D space is divided into quadrants

\( \langle \text{axes, 1st or positive quadrant} \rangle \) [Fig. 3]

3D space is divided into octants

\( \langle 3D \text{ axes, } xy\text{-plane, } yz\text{-plane, } xz\text{-plane, 1st or positive octant} \rangle \) [Fig. 4]

Example. Plot \( P(4,3,0) \) and \( Q(1,6, \text{ --} 4) \)

\( \langle \text{axes, points plotted in perspective} \rangle \) [Fig. 5]

A single equation in \( \mathbb{R}^2 \) represents a curve.
**Example** \( y = x^2 \)

\(< 0 \ldots x^-, 0 \ldots y^-, \text{parabola, } \{(x, x^2) | x \in \mathbb{R} \} > \) [Fig. 6]

A single equation in \( \mathbb{R}^3 \) represents a **surface**.

**Example** \( y = x^2 \)

\(< 0 \ldots x^-, 0 \ldots y^-, 0 \ldots z^-, \text{parabolic cylinder, } \{(x, x^2, z) | x, z \in \mathbb{R} \} > \) [Fig. 7]

**Distance in \( \mathbb{R}^2 \)**

\(< 2D \text{ axes, } 1^{\text{st}} \text{ quadrant, } P(x_1, y_1), Q(x_2, y_2), \text{ segment } |PQ|, \text{ complete right triangle with sides of length } |x_2 - x_1| \text{ and } |y_2 - y_1| > \) [Fig. 8]

Pythagorean theorem

\[ |PQ|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 \]

\[ = (x_2 - x_1)^2 + (y_2 - y_1)^2 \]

**Distance in \( \mathbb{R}^3 \)**

\(< 3D \text{ axes, show plane } z = z_1 \text{ as } xy\text{-plane, } P(x_1, y_1, z_1), Q(x_2, y_2, z_1), R(x_2, y_2, z_2), \text{ sides of right triangle} > \) [Fig. 9]

\[ |PQ|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \]

\[ |PR|^2 = |PQ|^2 + |QR|^2 \]

\[ = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \]

\[ |PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

**Distance between \( P_1(x_1, y_1, z_1) \) and \( P_2(x_2, y_2, z_2) \)**

\[ |P_1P_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \]

\[ |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

**Example.** Find the distance between \( P(4,3,0) \) and \( Q(1,6, -4) \)

\[ |PQ|^2 = (4 - 1)^2 + (3 - 6)^2 + (0 - (-4))^2 = 9 + 9 + 16 = 34 \]
\[ |PQ| = \sqrt{34} \quad \blacksquare \]

Circle of radius \( r \) centered at \((a, b)\) in 2D

\(\langle 2\text{D axes and 1}^{\text{st}}\text{ quadrant}, (a, b), \text{circle, radius } r \text{ to } (x, y) \text{ on circle} \rangle \) [Fig. 10]

Distance between \((x, y)\) and \((a, b)\) is \( r \)

\[ \sqrt{(x - a)^2 + (y - b)^2} = r \]
\[ (x - a)^2 + (y - b)^2 = r^2 \]

Equation for circle in 2D

Sphere of radius \( r \) centered at \((a, b, c)\) in 3D

\(\langle 3\text{D axes and 1}^{\text{st}}\text{ octant}, (a, b, c), \text{sphere, radius } r \text{ to } (x, y, z) \text{ on sphere} \rangle \) [Fig. 11]

Distance between \((x, y, z)\) and \((a, b, c)\) is \( r \)

\[ \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} = r \]
\[ (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \]

Equation for sphere of radius \( r \) and center \((a, b, c)\) in 3D

**Example.** Show that the given equation represents a sphere. Find its center and radius.

\[ 2x^2 + 2y^2 + 2z^2 + 4y - 2z = 1 \]
\[ x^2 + y^2 + z^2 + 2y - z = \frac{1}{2} \]
\[ x^2 + (y^2 + 2y + 1) - 1 + (z^2 - z + \frac{1}{4}) - \frac{1}{4} = \frac{1}{2} \]
\[ x^2 + (y + 1)^2 + (z - \frac{1}{2})^2 = 1 + \frac{1}{4} + \frac{1}{2} = \frac{7}{4} \]

Center \((0, -1, \frac{1}{2})\)

Radius \( \sqrt{\frac{7}{2}} \) \( \blacksquare \)
10.2 Vectors

**Motivation**
Quantities with magnitude and direction

Examples. Position, velocity, force.

**Algebraic Properties of Vectors**
In 2D space, vectors are ordered pairs of real numbers.

\[ \mathbf{a} = (a_1, a_2) \]

In 3D space, vectors are ordered triples

\[ \mathbf{a} = (a_1, a_2, a_3) \]

\( a_1 \) is the \( x \)-component

\( a_2 \) is the \( y \)-component

\( a_3 \) is the \( z \)-component

**Addition of vectors**
\[ \mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) \]

\[ = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \]

\[ \mathbf{a} - \mathbf{b} = (a_1, a_2, a_3) - (b_1, b_2, b_3) \]

\[ = (a_1 - b_1, a_2 - b_2, a_3 - b_3) \]

**Zero Vector**
\[ \mathbf{0} = (0,0,0) \]

\[ \mathbf{a} + \mathbf{0} = \mathbf{a} \]
Multiplication by Scalars

scalar $k$ (a real number)

\[ ka = k\langle a_1, a_2, a_3 \rangle = \langle ka_1, ka_2, ka_3 \rangle \]

Three Algebraic Properties

(1) commutative law

\[ \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \]

(2) associative law

\[ \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \]

(3) distributive law

\[ k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b} \]

Example. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{a} + 4\mathbf{b}$, where

$\mathbf{a} = \langle 3, 2, -1 \rangle$, $\mathbf{b} = \langle 0, 6, 7 \rangle$

?? $\mathbf{a} + \mathbf{b} = \langle 3, 8, 6 \rangle$

?? $\mathbf{a} - \mathbf{b} = \langle 3, -4, -8 \rangle$

?? $3\mathbf{a} + 4\mathbf{b} = \langle 9, 30, 25 \rangle$ ■

Geometric Properties of Vectors

In 2D, consider $\langle 2,1 \rangle$

\[ \langle 0 \ldots 6 \ldots x-, 0 \ldots 2 \ldots y-, P(2,1), A(4,1), B(6,2) \rangle \] [Fig. 12]

\[ \overrightarrow{OP} = \langle 2,1 \rangle \quad \text{position vector of point } P(2,1) \quad \text{(add)} \]

\[ \overrightarrow{AB} = \langle 2,1 \rangle \]

These are different representations of the vector $\langle 2,1 \rangle$.

Geometrically, vectors add tip-to-tail

\[ \langle 0, A(a_1, a_2), B(b_1, b_2), \overrightarrow{0A}, \overrightarrow{0B}, \overrightarrow{AB} \rangle \] [Fig. 13]
\[ \overrightarrow{0A} + \overrightarrow{AB} = \overrightarrow{0B} \]
\[ \overrightarrow{AB} = \overrightarrow{0B} - \overrightarrow{0A} \]
\[ = (b_1, b_2) - (a_1, a_2) \]
\[ = (b_1 - a_1, b_2 - a_2) \]
consistent with
\[ \overrightarrow{0A} + \overrightarrow{AB} = \overrightarrow{0B} \]
\[ \langle a_1, a_2 \rangle + \langle b_1 - a_1, b_2 - a_2 \rangle = \langle b_1, b_2 \rangle \]

**Three Dimensions**
\[ \overrightarrow{0A} = \langle a_1, a_2, a_3 \rangle \]
\[ \overrightarrow{0B} = \langle b_1, b_2, b_3 \rangle \]
\[ \overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle \]

**Example**
\[ \langle -4, 0, 2, -x, 0, 2, y, -a, b, c \rangle \] [Fig. 14]
\[ b = \langle 1, 2 \rangle, \quad a = \langle -3, 1 \rangle \]
what is c?
\[ a + c = b \quad \text{add tip-to-tail} \]
\[ c = b - a = \langle 1, 2 \rangle - \langle -3, 1 \rangle = \langle 4, 1 \rangle \]

**Magnitude or length of a vector**
In 2D
\[ a = \langle a_1, a_2 \rangle \]
\[ |a| = \sqrt{a_1^2 + a_2^2} \]
In 3D
\[ a = \langle a_1, a_2, a_3 \rangle \]
\[ |a| = \sqrt{a_1^2 + a_2^2 + a_3^2} \]
Two notations for components

Vectors:
components denoted by subscript: \( a = \langle a_1, a_2, a_3 \rangle \)
components denoted by base: \( x = \langle x_0, y_0, z_0 \rangle \)

Points:
components denoted by subscript: \( P(x_1, y_1, z_1) \)
components denoted by base: \( A(a_1, a_2, a_3) \)

Both notations are frequently seen

Example. Length of vector between two points in space
\[ \langle A(x_1, y_1, z_1), \ B(x_2, y_2, z_2), \ \overline{AB} \rangle \]
Let \( a = \overline{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \)
\[ |a| = |\overline{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

Multiplication by a scalar
\[ \langle \ldots - 2 \ldots 0 \ldots 4 \ldots x', \ldots - 1 \ldots 0 \ldots 2 \ldots y' \rangle \] [Fig. 15]
\( a = \langle 2, 1 \rangle \) \( \langle \text{add } a \rangle \)
\(-a = \langle -2, -1 \rangle \) \( \langle \text{add } -a \rangle \)
\( 2a = \langle 4, 2 \rangle \) \( \langle \text{add } 2a \rangle \)
\( a, -a, 2a \) are parallel.

Properties of Lengths
Let \( a, b \) be any two vectors and let \( k \) be a scalar
1. $|a| \geq 0$

   $|a| = 0$ if and only if $a = 0$.

2. $|ka| = |k||a|$

3. Triangle Inequality

   $|a + b| \leq |a| + |b|$

   (sketch $0, a, b, a + b$) [Fig. 16]

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**Unit vectors**

A unit vector is a vector of length 1

$\langle... 0 \ldots 1 \ldots x, \ldots 0 \ldots 1 \ldots y, \ldots \rangle, \hat{i} = \langle 1,0 \rangle, \hat{j} = \langle 0,1 \rangle, \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ [Fig. 17]

$$\left|\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

For any vector $v \neq 0$

$$\frac{v}{|v|}$$ is a unit vector

$$\left|\frac{v}{|v|}\right| = \frac{1}{|v|}|v| = 1$$

**Standard basis vectors**

In 2D $\hat{i} = \langle 1,0 \rangle$

$\hat{j} = \langle 0,1 \rangle$

In 3D $\hat{i} = \langle 1,0,0 \rangle$

$\hat{j} = \langle 0,1,0 \rangle$

$\hat{k} = \langle 0,0,1 \rangle$

Any vector can be represented in terms of $\hat{i}, \hat{j}, \hat{k}$

$\langle 2,1 \rangle = \langle 2,0 \rangle + \langle 0,1 \rangle$

$= 2\langle 1,0 \rangle + \langle 0,1 \rangle$

$= 2\hat{i} + \hat{j}$
\( \langle 5, 2, -4 \rangle = 5 \hat{i} + 2 \hat{j} - 4 \hat{k} \)

# 10.3 The Dot Product

The dot product is also known as the scalar or inner product

### 2D

\( \mathbf{a} = \langle a_1, a_2 \rangle \)
\( \mathbf{b} = \langle b_1, b_2 \rangle \)
\( \mathbf{a} \cdot \mathbf{b} = \langle a_1 b_1 + a_2 b_2 \rangle \)

### 3D

\( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \)
\( \mathbf{b} = \langle b_1, b_2, b_3 \rangle \)
\( \mathbf{a} \cdot \mathbf{b} = \langle a_1 b_1 + a_2 b_2 + a_3 b_3 \rangle \)

**Example.** \( \mathbf{a} = \langle -1, -2, -2 \rangle \)
\( \mathbf{b} = \langle 2, 8, -6 \rangle \)
\( \mathbf{a} \cdot \mathbf{b} = -2 - 16 + 12 = -6 \)

Properties of the dot product

1. \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \)
2. \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)
3. \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)
4. \( k(\mathbf{a} \cdot \mathbf{b}) = k\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot k\mathbf{b} \)
5. \( \mathbf{a} \cdot \mathbf{0} = 0 \)

Prove 1.

\[ |\mathbf{a}| = |\langle a_1, a_2, a_3 \rangle| \]
\[ = \sqrt{a_1^2 + a_2^2 + a_3^2} \]
\[ \mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2 \]  

The angle between two vectors is the smallest positive angle between the vectors drawn with their tails at the same point.

\[ \langle \text{sketch } \mathbf{a}, \mathbf{b}, \theta \rangle \text{ [Fig. 18]} \]

**Theorem.** If \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \) then

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta) \]

**Proof** \( \langle \text{sketch } \mathbf{a}, \mathbf{b}, \theta, \mathbf{b} - \mathbf{a} \rangle \) [Fig. 19]

Law of cosines

\[ |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}| |\mathbf{b}| \cos(\theta) \]

Properties of dot product

\[ (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2 \mathbf{a} \cdot \mathbf{b} \]

Thus

\[ |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}| |\mathbf{b}| \cos(\theta) = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2 \mathbf{a} \cdot \mathbf{b} \]

\[ -2|\mathbf{a}| |\mathbf{b}| \cos(\theta) = -2 \mathbf{a} \cdot \mathbf{b} \]

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta) \]  

**Example.** Find the angle between the vectors \( \langle 3, 1 \rangle \) and \( \langle 2, 4 \rangle \).

\[ \langle \text{sketch } 0 \ldots 3 \ldots x-, 0 \ldots 4 \ldots y-, \mathbf{a} = \langle 3, 1 \rangle, \mathbf{b} = \langle 2, 4 \rangle, \theta \rangle \text{ [Fig. 20]} \]

\[
\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \\
\langle 3, 1 \rangle \cdot \langle 2, 4 \rangle = 10 \\
|\langle 3, 1 \rangle| = \sqrt{10} \\
|\langle 2, 4 \rangle| = \sqrt{20} \\
\cos(\theta) = \frac{10}{\sqrt{10} \sqrt{20}} = \frac{1}{\sqrt{2}}
\]
\[ \theta = \pi/4 \]

Two vectors are orthogonal or perpendicular if the angle between them is \( \pi/2 \).

\[
\text{[Fig. 21]} 
\]

\[ \cos \left( \frac{\pi}{2} \right) = 0 \]

If \( \mathbf{a} \neq \mathbf{0} \) and \( \mathbf{b} \neq \mathbf{0} \) then \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal iff \( \mathbf{a} \cdot \mathbf{b} = 0 \)

**Example.** Determine whether the following vectors are orthogonal: \( \mathbf{a} = \langle -1, 5, 2 \rangle \) and \( \mathbf{b} = \langle 4, 2, -3 \rangle \).

\[ \mathbf{a} \cdot \mathbf{b} = -4 + 10 - 6 = 0 \text{ therefore the vectors are orthogonal.} \]

Define the component or scalar projection of a vector along another vector.

\[ \text{[Fig. 22]} \]

\[ \text{comp}_a \mathbf{b} = |\mathbf{b}| \cos(\theta) \]

Define the projection or vector projection of a vector along another vector.

\[ \text{[Fig. 23]} \]

\[ \text{proj}_a \mathbf{b} = |\mathbf{b}| \cos(\theta) \frac{\mathbf{a}}{|\mathbf{a}|} \]

**Example.** Find the scalar and vector projection of \( \mathbf{b} \) on to \( \mathbf{a} \): \( \mathbf{a} = \langle 3, 1 \rangle \), \( \mathbf{b} = \langle 2, 3 \rangle \).

Recall \( \cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \)

\[ \text{comp}_a \mathbf{b} = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{9}{\sqrt{10}} \]

\[ \text{proj}_a \mathbf{b} = \text{comp}_a \mathbf{b} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{9}{\sqrt{10}} \frac{\langle 3, 1 \rangle}{\sqrt{10}} = \frac{9}{10} \langle 3, 1 \rangle \]

**Example** Let
\[ \mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} = \langle 2, -3, 1 \rangle \]
\[ \mathbf{b} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k} = \langle 1, 6, -2 \rangle \]

Then
\[
\text{comp}_a \mathbf{b} = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{-18}{\sqrt{14}}
\]
\[
\text{proj}_a \mathbf{b} = \text{comp}_a \mathbf{b} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{-18}{\sqrt{14}} \left( \frac{2, -3, 1}{\sqrt{14}} \right) = \frac{-18}{14} \langle 2, -3, 1 \rangle \]

**Application to Work**

For force \( \mathbf{F} \) in the same direction as displacement \( \mathbf{D} \)

\[ W = \mathbf{F} \cdot \mathbf{D} \]

**Example.** Work sliding a crate on a flat surface

\langle \text{surface}, \text{crate}, \mathbf{F} = 50\text{N}, \mathbf{D} = 10\text{m} \rangle \quad [\text{Fig. 24}]

\[ W = \mathbf{F} \cdot \mathbf{D} = (50\text{N})(10\text{m}) = 500\text{J} \]

In US customary units

\langle \text{surface}, \text{crate}, \mathbf{F} = 10\text{ lbs}, \mathbf{D} = 30\text{ft} \rangle \quad [\text{Fig. 25}]

\[ W = 300 \text{ ft lbs} \]

What if displacement is not in the same direction as the force?

\langle \text{wagon with handle up at angle } \theta, \text{ horizontal displacement vector } \mathbf{D} \text{ and force } \mathbf{F} \text{ in direction of handle, projection of } \mathbf{F} \text{ on } \mathbf{D} \text{ of magnitude } |\mathbf{F}| \cos(\theta) \rangle \quad [\text{Fig. 26}]

Work done is the product of the component of \( \mathbf{F} \) along \( \mathbf{D} \) and the distanced moved

\[ W = (|\mathbf{F}| \cos(\theta)) |\mathbf{D}| = |\mathbf{F}| |\mathbf{D}| \cos(\theta) = \mathbf{F} \cdot \mathbf{D} \]

**Example.** Suppose \( |\mathbf{F}| = 50\text{N}, \theta = 45^\circ = \frac{\pi}{4} \text{, } |\mathbf{D}| = 10\text{m} \)

Then \[ W = (50\text{N})(10\text{m})(\cos \left( \frac{\pi}{4} \right)) = 500 \frac{\sqrt{2}}{2} \approx 354\text{J} \]

[\( \blacksquare \)]