Math 420/520

Assignment 4

1. Let \( \mathcal{V} = \mathbb{C}^n \), considered as a vector space over \( \mathbb{R} \). Prove that \( \dim(\mathcal{V}) = 2n \).

Let \( e_1, \ldots, e_n \) be the standard basis in \( \mathbb{R}^n \), and let \( g_j = ie_j \) for \( j = 1, \ldots, n \). I claim that the set of \( 2n \) vectors \( e_1, \ldots, e_n, g_1, \ldots, g_n \) is a basis for \( \mathcal{V} \), considered as a vector space over \( \mathbb{R} \). We have to show that these vectors are linearly independent and span \( \mathcal{V} \).

To see that they are linearly independent, suppose

\[
\sum_{j=1}^{n} \alpha_j e_j + \sum_{j=1}^{n} \beta_j g_j = 0,
\]

where \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \) are real numbers. Then

\[
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}
+ \beta_1
\begin{bmatrix}
i \\
0 \\
\vdots \\
i
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
0 \\
0 \\
\vdots \\
i
\end{bmatrix}
= 0,
\]

so

\[
\begin{bmatrix}
\alpha_1 + i\beta_1 \\
\alpha_2 + i\beta_2 \\
\vdots \\
\alpha_n + i\beta_n
\end{bmatrix}
= 0,
\]

so \( \alpha_j + i\beta_j = 0 \) for \( j = 1, \ldots, n \). This, in turn, implies \( \alpha_j = 0 \) and \( \beta_j = 0 \) for \( j = 1, \ldots, n \). Thus \( e_1, \ldots, e_n, g_1, \ldots, g_n \) are linearly independent.

To see that they span \( \mathcal{V} \), we must show that every element of \( \mathcal{V} \) can be written as a linear combination of \( e_1, \ldots, e_n, g_1, \ldots, g_n \) with real coefficients. Let \( x \) be an arbitrary element of \( \mathcal{V} \), say

\[
x = \begin{bmatrix}
\gamma_1 + i\delta_1 \\
\gamma_2 + i\delta_2 \\
\vdots \\
\gamma_n + i\delta_n
\end{bmatrix},
\]
where $\gamma_1$, \ldots, $\gamma_n$, $\delta_1$, \ldots, $\delta_n$ are real. Then $x = \gamma_1 e_1 + \cdots + \gamma_n e_n + \delta_1 g_1 + \cdots + \delta_n g_n$. Thus $x$ is a linear combination of $e_1$, \ldots, $e_n$, $g_1$, \ldots, $g_n$ with real coefficients. This shows that $e_1$, \ldots, $e_n$, $g_1$, \ldots, $g_n$ span $\mathcal{V}$.

Since this basis has $2n$ elements, the dimension of $\mathcal{V}$ is $2n$.

2. Let $\mathcal{V}$ be the set of real polynomials of degree $\leq 2$. Then $\mathcal{V}$ is a vector space over $\mathbb{R}$. Let $p_1(x) = 2$, $p_2(x) = 3 + 4x$, $p_3(x) = 5 + 6x + 7x^2$. Clearly $p_1$, $p_2$, and $p_3$ are all members of $\mathcal{V}$. Show that $\{p_1, p_2, p_3\}$ is a basis of $\mathcal{V}$.

**Sample Solution 1**: We established previously (in class) that $1$, $x$, $x^2$ form a basis for $\mathcal{V}$. Thus $\dim(\mathcal{V}) = 3$, so any linearly independent 3-element set in $\mathcal{V}$ is a basis (invoking Exercise 4, part a, to be proved below). Therefore we just have to show that $p_1$, $p_2$, $p_3$ are linearly independent. To this end, suppose

$$\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) = 0.$$ 

Then, after some elementary algebra,

$$(2\alpha_1 + 3\alpha_2 + 5\alpha_3) + (4\alpha_2 + 6\alpha_3)x + (7\alpha_3)x^2 = 0.$$ 

Since $1$, $x$, $x^2$ are linearly independent, we conclude that $2\alpha_1 + 3\alpha_2 + 5\alpha_3 = 0$, $4\alpha_2 + 6\alpha_3 = 0$, and $7\alpha_3 = 0$. The last equation implies $\alpha_3 = 0$. Substituting $\alpha_3 = 0$ into the second equation, we get $4\alpha_2 = 0$, which implies $\alpha_2 = 0$. Finally, substituting $\alpha_2 = 0$ and $\alpha_3 = 0$ into the first equation, we get $2\alpha_1 = 0$, which implies $\alpha_1 = 0$. Thus $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so $\{p_1, p_2, p_3\}$ is a linearly independent set.

**Sample Solution 2**: In class I stated the following result: $\{v_1, \ldots, v_k\}$ is a basis for $\mathcal{V}$ if and only if every $v \in \mathcal{V}$ can be written as a linear combination of $v_1, \ldots, v_k$ in exactly one way. I didn’t prove this result; I left it as an easy exercise for you. We can use this result to work this problem.

Let $p(x) = c_1 + c_2 x + c_3 x^2$ be an arbitrary element of $\mathcal{V}$. We must show that there exist unique $\alpha_1$, $\alpha_2$, and $\alpha_3 \in \mathbb{R}$ such that

$$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x). \quad (1)$$
We can rewrite this equation as

\[ c_1 + c_2x + c_3x^2 = (2\alpha_1 + 3\alpha_2 + 5\alpha_3) + (4\alpha_2 + 6\alpha_3)x + (7\alpha_3)x^2. \]

Since \( 1, x, x^2 \) are linearly independent, we have \( c_1 = 2\alpha_1 + 3\alpha_2 + 5\alpha_3, \)
\( c_2 = 4\alpha_2 + 6\alpha_3, \) and \( c_3 = 7\alpha_3. \) We can write these three linear equations
as a single matrix equation

\[
\begin{bmatrix}
2 & 3 & 5 \\
0 & 4 & 6 \\
0 & 0 & 7 \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{bmatrix} =
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\end{bmatrix}.
\]

(2)

In this equation, the coefficient matrix is upper triangular and has all
main-diagonal entries nonzero. Therefore (by Exercise 3 of Assignment
2), it is nonsingular. Thus the system (2) has a unique solution \( \alpha_1, \alpha_2, \alpha_3. \) Thus (1) has a unique solution.

3. Suppose \( v_1, \ldots, v_m \) is a spanning set for the vector space \( V. \) Prove that
\( v_1, \ldots, v_m \) has a subset that is a basis of \( V. \) You may give either an
informal (but careful and correct) argument that removes one vector
at a time or a formal induction proof.

Here's and informal argument: If \( v_1, \ldots, v_m \) are linearly independent,
we are done. Suppose they are linearly dependent. Then there is a
\( j \) such that \( v_j \in \text{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m\}. \) I claim that \( v_1, \ldots, \)
\( v_{j-1}, v_{j+1}, \ldots, v_m \) is a spanning set for \( V. \) To show this, we must show
that an arbitrary \( x \in V \) can be written as a linear combination of
\( v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m. \) To begin with, we know that

\[ x = \alpha_1 v_1 + \cdots + \alpha_j v_j + \cdots + \alpha_m v_m, \tag{3} \]

for some scalars \( \alpha_1, \ldots, \alpha_m, \) because \( v_1, \ldots, v_m \) span \( V. \) We also have

\[ v_j = \beta_1 v_1 + \cdots + \beta_{j-1} v_{j-1} + \beta_{j+1} v_{j+1} + \cdots + \beta_m v_m, \tag{4} \]

for some scalars \( \beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_m, \) because
\( v_j \in \text{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m\}. \)

Substituting (4) into (3), we obtain

\[
x = (\alpha_1 + \alpha_j \beta_1) v_1 + \cdots + (\alpha_{j-1} + \alpha_j \beta_{j-1}) v_{j-1} \\
+ (\alpha_{j+1} + \alpha_j \beta_{j+1}) v_{j+1} + \cdots + (\alpha_m + \alpha_j \beta_m) v_m \\
= \gamma_1 v_1 + \cdots + \gamma_{j-1} v_{j-1} + \gamma_{j+1} v_{j+1} + \cdots + \gamma_m v_m,
\]

where \( \gamma_j = \alpha_j + \alpha_{j+1} \beta_{j+1} + \cdots + \alpha_m \beta_m. \)
where \( \gamma_k = \alpha_k + \alpha_j \beta_k \). This proves that \( v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m \) span \( \mathcal{V} \).

Now, if \( v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m \) are linearly independent, we are done. If not, we can remove another vector and still have a spanning set, as just shown. We can continue the process until we obtain a basis for \( \mathcal{V} \).

4. Let \( \mathcal{V} \) be a vector space of dimension \( k \). Prove the following two statements.

(a) Any set of \( k \) linearly independent vectors in \( \mathcal{V} \) is a basis for \( \mathcal{V} \).

Let’s prove this by contradiction. Suppose we have a set of \( k \) linearly independent vectors that is not a basis. We proved in class that this set (i.e., any linearly independent set) can be extended to make a basis of \( \mathcal{V} \). This basis necessarily has more than \( k \) vectors, so the dimension of \( \mathcal{V} \) is greater than \( k \). This contradicts the basic assumption that \( \dim(\mathcal{V}) = k \).

(b) Any spanning set of \( k \) vectors in \( \mathcal{V} \) is a basis for \( \mathcal{V} \).

Let’s prove this by contradiction, too. Suppose we have a spanning set of \( k \) vectors in \( \mathcal{V} \) that is not a basis for \( \mathcal{V} \). Then, by the previous problem, this spanning set has a subset, which must be a proper subset, that is a basis for \( \mathcal{V} \). Thus \( \dim(\mathcal{V}) < k \). This contradicts the assumption that \( \dim(\mathcal{V}) = k \).

5. Prove that if \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are both subspaces of \( \mathcal{V} \), then \( \mathcal{S}_1 \cap \mathcal{S}_2 \) is also a subspace of \( \mathcal{V} \).

We begin by showing that \( \mathcal{S}_1 \cap \mathcal{S}_2 \) is nonempty. We know that \( 0 \in \mathcal{S}_1 \) and \( 0 \in \mathcal{S}_2 \). Therefore \( 0 \in \mathcal{S}_1 \cap \mathcal{S}_2 \).

Now we show that \( \mathcal{S}_1 \cap \mathcal{S}_2 \) is closed under vector addition. Suppose \( v, w \in \mathcal{S}_1 \cap \mathcal{S}_2 \). Then \( v, w \in \mathcal{S}_j \), for \( j = 1, 2 \). Since \( \mathcal{S}_j \) is a subspace, it is closed under vector addition, so we conclude that \( v + w \in \mathcal{S}_j \) for \( j = 1, 2 \). Thus \( v + w \in \mathcal{S}_1 \cap \mathcal{S}_2 \).

Finally, we show that \( \mathcal{S}_1 \cap \mathcal{S}_2 \) is closed under scalar multiplication. Suppose \( v \in \mathcal{S}_1 \cap \mathcal{S}_2 \) and \( \alpha \in \mathbb{F} \). Then \( v \in \mathcal{S}_j \), for \( j = 1, 2 \). Since \( \mathcal{S}_j \) is a subspace, it is closed under scalar multiplication, so we conclude that \( \alpha v \in \mathcal{S}_j \) for \( j = 1, 2 \). Thus \( \alpha v \in \mathcal{S}_1 \cap \mathcal{S}_2 \).
Since $\mathcal{S}_1 \cap \mathcal{S}_2$ is nonempty and closed under vector addition and scalar multiplication, it is a subspace of $\mathcal{V}$. 