Math 420/520

Assignment 10  due December 14, 2001

1. Let $A \in \mathbb{C}^{n \times n}$.

   (a) Show that if $A$ is nonsingular and $(\lambda, v)$ is an eigenpair of $A$, then
       $(\lambda^{-1}, v)$ is an eigenpair of $A^{-1}$.

   (b) Show that if $(\lambda, v)$ is an eigenpair of $A$, then $(\lambda^j, v)$ is an eigenpair
       of $A^j$ for every nonnegative integer $j$. (Combining this result with that of part (a),
       we see that the result holds for negative integers as well.)

   (c) If $q$ is a polynomial, say $q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$, define
       the matrix $q(A)$ by

       $$q(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_k A^k.$$  

       Show that if $(\lambda, v)$ is an eigenpair of $A$, then $(q(\lambda), v)$ is an eigen-
       pair of $q(A)$.

2. Recall that a matrix $P \in \mathbb{C}^{n \times n}$ is idempotent if $P^2 = P$. (Idempotents are
   projectors.) Show that if $P$ is idempotent, $P \neq 0$, and $P \neq I$, then $\sigma(P) = \{0, 1\}$.
   Characterize the eigenspaces of $P$. Show that $P$ is semisimple.

3. A matrix $N \in \mathbb{C}^{n \times n}$ is called nilpotent if there is a positive integer $k$
   such that $N^k = 0$.

   (a) Show that if $N$ is nilpotent, $\sigma(N) = \{0\}$.

   (b) Show that if $N$ is nilpotent, then $N$ is unitarily similar to a strictly
       upper triangular matrix.
4. Recall that if $A \in \mathbb{C}^{n \times n}$, $\text{tr}(A) = \sum_{j=1}^{n} a_{jj}$.

(a) Show that if $C \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}^{n \times m}$, then

$$\text{tr}(CD) = \text{tr}(DC).$$

Notice that $CD$ and $DC$ are square, even though $C$ and $D$ may not be.

(b) Use the result of part (a) (square case) in an intelligent way to show that if $B = S^{-1}AS$, then $\text{tr}(B) = \text{tr}(A)$.

(c) Show that if $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\text{tr}(A) = \lambda_1 + \cdots + \lambda_n.$$