Math 420/520

Solutions to Exam 2

1. Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$, and let $v_1, v_2, \ldots, v_k \in \mathcal{V}$.

(a) (5 points) State clearly and concisely what is meant by linear independence of $v_1, v_2, \ldots, v_k$.

The only solution of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$ with $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{F}$ is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

(b) (5 points) $v_1, v_2, \ldots, v_k$ is a spanning set for $\mathcal{V}$ if $\mathcal{V} = \text{span}\{v_1, \ldots, v_k\}$. Exactly what does this mean?

For every $v \in \mathcal{V}$ there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_k v_k$.

(c) (5 points) What is a basis for a vector space $\mathcal{V}$?

A basis is a linearly independent spanning set for $\mathcal{V}$.

(d) (5 points) What do we mean by the dimension of a vector space?

The dimension of a vector space is the number of elements in a basis for the space.

(e) (5 points) Before we could define dimension, we had to prove a key theorem. What does that theorem say?

Any two bases for a space have the same number of elements.
2. (5 points) Let \( \mathcal{V} \) be a vector space over \( \mathbb{F} \), and let \( \mathcal{B} = \{v_1, v_2\} \) be a basis for \( \mathcal{V} \). Suppose \( U, S, T_1, \) and \( T_2 \in \mathcal{L}(\mathcal{V}, \mathcal{V}) \) are linear transformations related by \( U = S(T_1 + T_2) \). Determine \([U]_{\mathcal{B}, \mathcal{B}}\), given that

\[
[S]_{\mathcal{B}, \mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad [T_1]_{\mathcal{B}, \mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad [T_2]_{\mathcal{B}, \mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

\[
[U]_{\mathcal{B}, \mathcal{B}} = [S]_{\mathcal{B}, \mathcal{B}}([T_1]_{\mathcal{B}, \mathcal{B}} + [T_2]_{\mathcal{B}, \mathcal{B}})
\]

\[
= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}.
\]

3. (10 points) Let \( T \in \mathcal{L}(\mathcal{V}, \mathcal{W}) \). Define \( \mathcal{N}(T) \), the null space of \( T \), and show that \( \mathcal{N}(T) \) is a subspace of \( \mathcal{V} \).

\[
\mathcal{N}(T) = \{v \in \mathcal{V} | T(v) = 0\}.
\]

To show that \( \mathcal{N}(T) \) is a subspace, we have to show that it is nonempty, closed under vector addition, and closed under scalar multiplication.

Since \( T(0) = 0 \), we see that \( 0 \in \mathcal{N}(T) \), so \( \mathcal{N}(T) \) is nonempty.

Now let \( v_1, v_2 \in \mathcal{N}(T) \). Then \( T(v_1) = 0 \) and \( T(v_2) = 0 \), so \( T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0 \). Therefore \( v_1 + v_2 \in \mathcal{N}(T) \). This shows that \( \mathcal{N}(T) \) is closed under vector addition.

Finally, let \( v \in \mathcal{N}(T) \) and \( \alpha \in \mathbb{F} \). Since \( v \in \mathcal{N}(T) \), we have \( T(v) = 0 \), so \( T(\alpha v) = \alpha T(v) = \alpha 0 = 0 \). Thus \( \alpha v \in \mathcal{N}(T) \). This shows that \( \mathcal{N}(T) \) is closed under scalar multiplication.
4. (10 points) Using the defining properties (axioms) of an inner product, show that in any inner product space $\mathcal{V}$ over $\mathbb{C}$,

$$\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$$

for all $x, y \in \mathcal{V}$ and all $\alpha \in \mathbb{C}$.

\[
\begin{align*}
\langle x, \alpha y \rangle & = \overline{\langle \alpha y, x \rangle} \quad \text{(conjugate symmetry)} \\
& = \overline{\alpha \langle y, x \rangle} \quad \text{(linearity in first argument)} \\
& = \overline{\alpha} \langle y, x \rangle \quad \text{(property of complex numbers)} \\
& = \overline{\alpha} \langle x, y \rangle \quad \text{(conjugate symmetry)}
\end{align*}
\]

5. (10 points) Show that if $\mathcal{V}$ is an inner product space, $x, y \in \mathcal{V}$, and $x \perp y$, then

$$\| x - y \|^2 = \| x \|^2 + \| y \|^2.$$

Since $x \perp y$, we have $\langle x, y \rangle = 0$ and $\langle y, x \rangle = 0$. Therefore

\[
\begin{align*}
\| x - y \|^2 & = \langle x - y, x - y \rangle \\
& = \langle x, x - y \rangle - \langle y, x - y \rangle \\
& = \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\
& = \| x \|^2 - 0 - 0 + \| y \|^2 \\
& = \| x \|^2 + \| y \|^2.
\end{align*}
\]
6. (10 points) Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces over $\mathbb{C}$, and let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Given $y \in \mathcal{W}$, define $L : \mathcal{V} \to \mathbb{C}$ by $Lx = \langle Tx, y \rangle$ for all $x \in \mathcal{V}$. Show that $L \in \mathcal{L}(\mathcal{V}, \mathbb{C})$.

Let $v_1, v_2 \in \mathcal{V}$. Then

\[
L(v_1 + v_2) = \langle T(v_1 + v_2), y \rangle \\
= \langle T(v_1) + T(v_2), y \rangle \quad \text{(linearity of } T) \\
= \langle T(v_1), y \rangle + \langle T(v_2), y \rangle \quad \text{(linearity of inner product)} \\
= L(v_1) + L(v_2).
\]

Let $v \in \mathcal{V}$ and $\alpha \in \mathbb{F}$. Then

\[
L(\alpha v) = \langle T(\alpha v), y \rangle \\
= \langle \alpha T(v), y \rangle \quad \text{(linearity of } T) \\
= \alpha \langle T(v), y \rangle \quad \text{(linearity of inner product)} \\
= \alpha L(v).
\]

Therefore $L$ is linear.

7. (10 points) Let $\mathcal{V}$ be an inner product space, and let $x, y \in \mathcal{V}$ with $y \neq 0$. Show that there is a unique $\beta \in \mathbb{F}$ such that $x - \beta y \perp y$. Give a formula for $\beta$.

$x - \beta y \perp y$ if and only if $\langle x - \beta y, y \rangle = 0$ if and only if $\langle x, y \rangle - \beta \langle y, y \rangle = 0$ if and only if

\[
\beta = \frac{\langle x, y \rangle}{\langle y, y \rangle}.
\]

(Since $y \neq 0$, it is guaranteed that $\langle y, y \rangle \neq 0$.)
8. (10 points) Let

\[ f_1(x) = e^x, \quad f_2(x) = xe^x, \quad f_3(x) = x^2e^x, \]

and let \( \mathcal{W} \) be the three-dimensional vector space span\( \{f_1, f_2, f_3\} \). Define \( T \in \mathcal{L}(\mathcal{W}, \mathcal{W}) \) by \( Tf = f' \) (first derivative). Compute \([T]_B,B\), the matrix of \( T \) with respect to the basis \( B = \{f_1, f_2, f_3\} \).

\[
\begin{align*}
Tf_1 &= 1f_1 + 0f_2 + 0f_3 \\
Tf_2 &= 1f_1 + 1f_2 + 0f_3 \\
Tf_3 &= 0f_1 + 2f_2 + 1f_3
\end{align*}
\]

Therefore

\[
[T]_B,B = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}.
\]
9. (10 points) Let \( T \in \mathcal{L}(V, W) \), where \( \dim(V) = n \). Define the rank and nullity of \( T \), and prove that rank plus nullity equals \( n \). (If you’re out of time, just sketch the proof; that is, indicate how you would prove it, leaving out the details that you don’t have time for.)

The rank of \( T \) is the dimension of the range of \( T \). The nullity of \( T \) is the dimension of the nullspace of \( T \).

Let \( k \) denote the nullity of \( T \), that is, \( k = \dim(\mathcal{N}(T)) \). Let \( v_1, \ldots, v_k \) be a basis for \( \mathcal{N}(T) \). We can extend this to a basis for \( V \). That is, there exist \( v_{k+1}, \ldots, v_n \) such that \( v_1, \ldots, v_n \) is a basis for \( V \).

I claim that \( Tv_{k+1}, \ldots, Tv_n \) is a basis for \( \mathcal{R}(T) \). If so, we have \( \text{rank}(T) = \dim(\mathcal{R}(T)) = n - k \), so rank plus nullity equals \( n \), as claimed.

To show that \( Tv_{k+1}, \ldots, Tv_n \) is a basis for \( \mathcal{R}(T) \), we first show that these vectors span \( \mathcal{R}(T) \). Let \( w \in \mathcal{R}(T) \). Then there is a \( v \in V \) such that \( Tv = w \). Now \( v = \alpha_1 v_1 + \cdots + \alpha_n v_n \) for some \( \alpha_1, \ldots, \alpha_n \in \mathbb{F} \), so

\[
    w = Tv = \alpha_1 Tv_1 + \cdots + \alpha_k Tv_k + \alpha_{k+1} Tv_{k+1} + \cdots + \alpha_n Tv_n \\
    = 0 + \cdots + 0 + \alpha_{k+1} Tv_{k+1} + \cdots + \alpha_n Tv_n \\
    = \alpha_{k+1} Tv_{k+1} + \cdots + \alpha_n Tv_n.
\]

Therefore \( Tv_{k+1}, \ldots, Tv_n \) span \( \mathcal{R}(T) \).

We now show that \( Tv_{k+1}, \ldots, Tv_n \) are linearly independent. Suppose

\[
    \alpha_{k+1} Tv_{k+1} + \cdots + \alpha_n Tv_n = 0.
\]

Then \( T(\alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n) = 0 \), so \( \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n \in \mathcal{N}(T) \).

Therefore \( \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_k v_k \) for some \( \alpha_1, \ldots, \alpha_k \in \mathbb{F} \). Thus, letting \( \alpha_j = -\beta_j \) for \( j = 1, \ldots, k \), we have

\[
    \alpha_1 v_1 + \cdots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n = 0.
\]

Since \( v_1, \ldots, v_n \) are linearly independent, we have \( \alpha_j = 0 \) for \( j = 1, \ldots, n \). In particular \( \alpha_{k+1} = \cdots = \alpha_n = 0 \), showing that \( Tv_{k+1}, \ldots, Tv_n \) are linearly independent.