Let $f(x) = x^2 + 6x + 4$. Applying the quadratic formula, we find that the equation $f(x) = 0$ has two real roots, $-3 \pm \sqrt{5}$. There is exactly one root in the interval $[-1, 0]$, namely $-3 + \sqrt{5} = 4/(-3-\sqrt{5}) \approx -0.7639$. Even though we already know the exact location of this root, it is instructive to locate it by numerical methods. In Exercises 1, 2, 4, and 5 you are asked to do this by various methods.

1. Use the bisection method to locate the solution of $f(x) = 0$ in $[-1, 0]$ with an error less than $1/128$. Give an interval that is known to contain the root.

2. (a) Show that the equation $f(x) = x^2 + 6x + 4 = 0$ ($x < 0$) can be rewritten in each of the forms

$$x = g(x) = \frac{-4}{x + 6}, \quad (x \neq -6)$$

and

$$x = h(x) = -\sqrt{-6x - 4}, \quad (x < -2/3).$$

(b) Apply functional iteration to each of these equations, starting with $x_0 = -5.2$ for the equation $x = g(x)$ and $x_0 = -0.77$ for the equation $x = h(x)$. Iterate until your answers converge to three or four decimal places.

(c) Differentiate the functions $g$ and $h$, and calculate

$$g'(-0.764), \quad g'(-5.236), \quad h'(-0.764), \quad \text{and} \quad h'(-5.236).$$

How do these numbers relate to the results of part (b)?

3. Suppose a sequence $(x_n)$ converges to $s$, and the convergence is of order $k \geq 1$. Show that

$$\lim_{n \to \infty} \frac{\log |x_{n+1} - s|}{\log |x_n - s|} = k.$$ 

This gives us a means of estimating the convergence rate of a sequence: Once $x_n$ is close to $x$, the ratio

$$\frac{\log |x_{n+1} - s|}{\log |x_n - s|}$$

approximates $k$. 

4. (a) Use the secant method to calculate the solution of \( f(x) = 0 \) to sixteen decimal places accuracy. Use starting values \( x_0 = 0 \) and \( x_1 = -1 \). (Using MATLAB, \texttt{format long e} will show sixteen digits.) Print out your iterates and check how many correct digits each one has. 

(b) Using your iterates \( x_n \) and known solution \( s \) (i.e. its sixteen-digit approximation), compute the ratios (1) for as many \( n \) as you can. (You don’t have to do this by hand; you can write MATLAB code to do it for you.) Based on these estimates, what does the convergence rate of the secant method seem to be? Is it close to the theoretical ratio given (without proof) in class?

5. (a) Use Newton’s method to calculate the solution of \( f(x) = 0 \) to sixteen decimal places accuracy. Use the starting values (i) \( x_0 = 0 \), (ii) \( x_0 = 10 \), and (iii) \( x_0 = -10 \). (Note the sudden convergence.) 

What happened in part (iii)? (b) Using your iterates \( x_n \) from part (i), compute the ratios (1) for as many \( n \) as you can. Are these ratios close to what you would expect, based on theory?

6. Newton’s method normally converges rapidly, once the iterates are sufficiently close to the solution. This exercise shows that it can sometimes take many iterations to get to the region of rapid convergence, if the initial guess is poor. Apply Newton’s method to the function \( k(x) = x^{20} - 10 \) to compute \( 20\sqrt{10} \). Start with the (poor) initial guess \( x_0 = 8 \). Notice that many iterations are needed before the iterates get close to the solution. However, once they get close enough, the convergence is rapid.

7. The equation \( x^2 - 6x + 9 = 0 \) has a solution \( x = 3 \). Apply Newton’s method to this equation, starting with \( x_0 = 4 \). Do five or more iterations and observe the results. What rate of convergence do you observe here? Is this what you expected? Explain what you see.

8. Suppose the function \( g \) satisfies \( g(2) > 2 \), \( g(3) < 3 \), \( |g'(x)| < .8 \) for all \( x \in [2, 3] \), and \( g'(s) = .7 \). Suppose we want to solve the equation \( x = g(x) \) to eight decimal places accuracy. Which would converge more rapidly, functional iteration applied to \( g \), or the bisection method applied to \( f(x) = x - g(x) \)? Estimate how many iterations will be needed for each method.
9. Each of the following sequences converges to zero. In each case, determine whether the sequence converges slower than linearly, linearly, quadratically, or cubically. Work directly from the definitions; do not use the crude tool developed in Exercise 3.

(a) $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}, 10^{-32}, \ldots$
(b) $.9, .81, .729, 0.9^4, 0.9^5, \ldots$
(c) $10^{-1}, 10^{-3}, 10^{-9}, 10^{-27}, 10^{-81}, \ldots$
(d) $1, 1/3, 1/5, 1/7, 1/9, \ldots$
(e) $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}, \ldots$

Quadratic and cubic convergence are qualitatively better than linear convergence. However, linear convergence can be quite satisfactory if the contraction number is small enough.