In this assignment you will analyze the convergence rate of the secant method for solving an equation \( f(x) = 0 \). To this end we will make use of first and second order divided differences of \( f \), which are defined as follows. If \( x, y, \) and \( z \) are three distinct points, first-order divided differences are given by

\[
f[x, y] = \frac{f(x) - f(y)}{x - y},
\]

and second-order divided differences are given by

\[
f[x, y, z] = \frac{f[x, y] - f[y, z]}{x - z}.
\]

Suppose \( f \) is twice continuously differentiable. By the mean value theorem, there is a \( c \) between \( x \) and \( y \) such that \( f[x, y] = f'(c) \). Later on we show in class that \( f[x, y, z] = \frac{1}{2}f''(d) \), where \( d \) is some (unknown) point in the smallest interval containing \( x, y, \) and \( z \).

1. Suppose we apply the secant method, which we can write as

\[
x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n]},
\]

and we obtain \( x_n \to s \), a solution of the equation of \( f(x) = 0 \). Show that the secant equation (1) is equivalent to

\[
x_{n+1} - s = (x_n - s)(x_{n-1} - s) \frac{f[x_{n-1}, x_n, s]}{f[x_{n-1}, x_n]}.\]  

(You may find it easiest to start with (2) and work toward (1).)

2. Show that if \( f'(s) \neq 0 \) and \( f''(s) \neq 0 \), then there is a nonzero constant \( M \) such that

\[
\lim_{n \to \infty} \frac{x_{n+1} - s}{(x_n - s)(x_{n-1} - s)} = M.
\]

3. Assume the secant method is convergent of order \( k \), i.e.

\[
\lim_{n \to \infty} \frac{x_{n+1} - s}{(x_n - s)^k} = C
\]
for some nonzero $C$ and some $k \geq 1$. We wish to determine $k$. Using (3) and (4) together, show $k^2 - k - 1 = 0$. Conclude that $k = (1 + \sqrt{5})/2 \approx 1.62$.

This argument is not quite airtight. We have not shown that the secant method is convergent of order $k$. We have shown that if it is convergent of order $k$ for some $k$, then that number $k$ must equal $(1 + \sqrt{5})/2$. 