In all of the following problems we are interested in solving the initial value problem
\[ y' = f(t, y), \quad y(t_0) = y_0 \]
on some interval \([a, b]\), with \(a = t_0\). A few simple methods for approximating the solution are given below:

**Euler:**
\[ y_{k+1} = y_k + h f(t_k, y_k) \]

**Backward Euler:**
\[ y_{k+1} = y_k + h f(t_{k+1}, y_{k+1}) \]

**Modified Euler:**
\[ y_{k+1} = y_k + h f(t_k + h/2, y_k + (h/2) f(t_k, y_k)) \]

**Heun:**
\[ y_{k+1} = y_k + \left(\frac{h}{2}\right) \left[ f(t_k, y_k) + f(t_{k+1}, y_k + h f(t_k, y_k)) \right] \]

**Trapezoidal:**
\[ y_{k+1} = y_k + \left(\frac{h}{2}\right) \left[ f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \right] \]

1. Use Taylor expansions of \(y(t_{k+1})\) and \(y'(t_{k+1})\) about the point \(t_k\) to show that the local truncation error of the trapezoidal method is \(O(h^3)\). Thus the trapezoidal method is of second order.

2. Write a computer program or programs in MATLAB to solve numerically the initial value problem
\[ y' = e^{-t} - 2y, \quad y(0) = 1, \]
on \([0, 1]\), with step size \(h = 0.1\), using each of the following methods:

(a) Euler
(b) Heun
(c) Backward Euler
(d) Trapezoidal

Methods (c) and (d) are implicit methods, which means that \(y_{k+1}\) appears on both sides of the equation. In general it is necessary to use functional iteration, Newton’s method, or some other numerical method to solve for \(y_{k+1}\). But in this case, since the DE is linear, it is possible to write an explicit formula for \(y_{k+1}\). Compare your results
with the exact solution \( y = e^{-t} \) and note that the second-order methods are more accurate than the first-order methods, and the implicit method of each order is slightly more accurate than the explicit method of the same order.

3. Repeat the previous problem with the initial value problem

\[
y' = 99e^{-t} - 100y, \quad y(0) = 1,
\]

whose exact solution is also \( y = e^{-t} \). This is an example of a stiff DE. The general solution \( e^{-t} + ce^{-100t} \) has a rapidly decaying term. Note the disastrous results from the explicit methods. (All explicit methods perform poorly on stiff DE’s; some implicit methods perform well. The latter are called stiffly stable. The backward Euler and trapezoidal methods are stiffly stable.)

4. Continuing the previous problem, decrease the stepsize by factors of 2 until you get good results with the explicit methods. Also, try the larger step sizes \( h = 0.5 \) and \( h = 1.0 \) with the implicit methods.