1. Associated with our model problem \(-u'' + bu' + cu = f\) is a linear differential operator \(Lu = -u'' + bu' + cu\). Let us restrict ourselves to the case \(b = 0\) and consider the operator \(L\) given by

\[
Lu = -u'' + cu
\]

with domain consisting of all functions in \(C^2[0, 1]\) satisfying the boundary conditions

\[
u(0) = 0 = u(1)
\]

(a) Compute all of the eigenfunctions and eigenvalues of \(L\). (It can be shown that all of the eigenvalues are real and satisfy \(\lambda > c\). You may use this fact. As an exercise, you might like to prove it.)

(b) Compare the eigenfunctions of \(L\) with the eigenvectors of \(A_h\), which were computed in class.

(c) How well do the smallest eigenvalues of \(A_h\) approximate the smallest eigenvalues of \(L\)?

2. Consider the convection-diffusion boundary value problem

\[-u''(x) + 10u'(x) = 0 \quad u(0) = 0, \quad u(1) = 1.\]

(a) Calculate four different numerical solutions to this problem by using both centered and upstream finite differences with \(h = 1/4\) and \(h = 1/8\). You may either set up systems of equations and solve them or use the analytical formulas that we derived in class.

(b) Make a plot or plots that compare these numerical solutions with the exact solution.

(c) Comment on your results.

3. Consider the convection-diffusion equation

\[-u''(x) + bu'(x) = 0.\]

Show that if we approximate the first derivative term by a first-order upstream finite difference, we get the same result as if we had made a centered difference approximation in the differential equation

\[-Du''(x) + bu'(x) = 0\]
for a certain choice of $D > 1$. Since $D$ is the diffusivity, this result can be interpreted as follows: Making a first-order upstream finite difference is equivalent to adding “artificial diffusion” to the system.